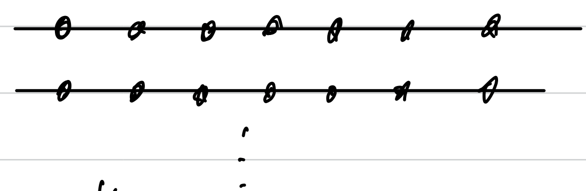
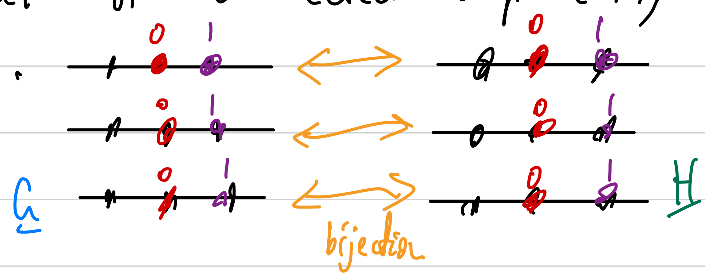
Math Logic: Model Theory & Computability
Lecture 17

Examples (continued). (c) Let $\sigma_{\text{gph}} := (E)$ be the signature of graphs. Let T_2 be the theory of all 2-regular undirected acyclic graphs, i.e. $T_2 := \{\varphi, \psi_2\} \cup \{\theta_n : n \geq 3\}$, where φ says the graph is undirected, ψ_2 says each vertex has exactly 2 neighbours, and θ_n says there is no cycle of length n. Let's completely characterize all models of T_2.

Obs. $\underline{G} := (V, E) \models T_2$ if and only if each of its connected components is a 2-regular tree, i.e. a $\mathbb{Z}$-line.



Let $\sim_{\underline{G}}$ denote the connectedness equivalence relation on V, i.e. $u \sim_{\underline{G}} v$ if u, v are in the same connected component. Two models $\underline{G}, \underline{H} \models T_2$ are isomorphic iff and if they have the same "number" of connected components, more precisely, $|V^{\underline{G}}/\sim_{\underline{G}}| = |V^{\underline{H}}/\sim_{\underline{H}}|$.



Prop. T_2 is κ-categorical for all uncftbl cardinals κ.

Proof. Let $\underline{G}, \underline{H} \models T_2$ of cardinality κ.
Then $V^{\underline{G}} = \bigsqcup_{C \in V^{\underline{G}}/\sim_{\underline{G}}} C$, so $\kappa = |V^{\underline{G}}| = |V^{\underline{G}}/\sim_{\underline{G}}| \cdot |\mathbb{Z}| = \max(|V^{\underline{G}}/\sim_{\underline{G}}|, \aleph_0)$,

hence $|V^{\underline{G}}/\sim_{\underline{G}}| = \kappa$, so $\underline{G}$ has κ-many connected components.
Same holds for $\underline{H}$, hence $\underline{G} \cong \underline{H}$. $\square$

(d) Let $\sigma_{\mathbb{Q}} :=$ the signature of $\mathbb{Q}$-vector space $:= (+, \lambda_q : q \in \mathbb{Q})$, where λ_q is a unary operation symbol. Let $VS_{\mathbb{Q}}$ be the $\sigma_{\mathbb{Q}}$-theory of vector spaces over $\mathbb{Q}$, which is infinite hence the axioms about the elements of $\mathbb{Q}$ need to be treated for each finite set of elements separately, e.g. to express that $\forall q_1, q_2 \in \mathbb{Q}$, $\forall$ vectors v, $(q_1 \cdot q_2) \cdot v = q_1 \cdot (q_2 \cdot v)$, we need to write one sentence for each pair $(q_1, q_2) \in \mathbb{Q}$, namely, $\varphi_{(q_1,q_2)} := \forall v \; \lambda_{q_1 q_2}(v) = \lambda_{q_1}(\lambda_{q_2}(v))$.

Thus, the models of VS are exactly the vector spaces over $\mathbb{Q}$. As we know, two $\mathbb{Q}$-vector spaces U, V are isomorphic iff they admit equinumerous bases B_U and B_V, i.e. $|B_U| = |B_V|$.

Prop. $VS_{\mathbb{Q}}$ is κ-categorical for every unctbl cardinal κ.

Proof. For a $\mathbb{Q}$-vector space V of cardinality κ, letting B_V denote a basis for V, we see that $\kappa = |V| = \left|\bigcup_{B \in \mathcal{P}_{fin}(B_V)} B \times \mathbb{Q}\right| = |\aleph_0| \cdot |\mathcal{P}_{fin}(B_V)|$
$= |\aleph_0| \cdot \left|\bigsqcup_{n \in \mathbb{N}} |B_V^n|\right| = |\aleph_0| \cdot |B_V| =$

$= \max(\aleph_0, |B_V|)$, hence $|B_V| = \kappa$. Thus, any two such vector spaces have equinumerous bases, are therefore isomorphic. $\square$

(e) For p prime or 0, recall the theory ACF_p of algebraically closed fields of characteristic p.

Lemma. For every alg. closed field F of characteristic p, there is a maximal transcendental set B (by Zorn) over the prime subfield $F_0 \subseteq F$ (i.e. the field generated by 1). Moreover, $|F| = \max(\aleph_0, |B|)$. Furthermore, two such fields are isomorphic iff their transcendence bases are equinumerous.

Proof-sketch. Let $B \subseteq F$ be a maximal trancendental set, i.e. each $b \in B$ is transcendental over $F_0(B \setminus \{b\}) :=$ the subfield generated by $B \setminus \{b\}$.

(In other words B is algebraically independent over F_0.) Then, any $a \in F$ is algebraic over $F_0(B)$ by the maximality of B. (If a is indep from $\{b_1, \dots, b_n\}$ then it can't be that b_1 is algebraic over $\{a, b_2, \dots, b_n\}$. This needs to be proved, but we'll skip it.)
Then, F is the algebraic closure of $F_0(B)$, and hence has cardinality $|\aleph_0| \cdot |B|$. Moreover, any bijection between transcedence bases of two fields $F_1, F_2 \models ACF_p$ is extended to an isomorphism (which may not be unique). $\boxtimes$

Cor. For any p prime or 0, ACF_p is κ-categorical for any unctbl cardinal κ.

Proof. Let $F_1, F_2 \models ACF_p$ of cardinality κ. Let B_1, B_2 be transcendence bases for F_1 and F_2. Then $\kappa = |F_i| = \max(\aleph_0, |B_i|)$ implies $|B_i| = \kappa$, so $|B_1| = |B_2|$, thus $F_1 \cong F_2$. $\square$

It turns out that the fact that in Examples (c)−(e) we had κ-categoricity for all unctb κ is not a mere coincidence:

Morley theorem. Let σ be a ctbl signature and T a σ-theory. If T is λ-categorical for some unctbl cardinal λ, then it is κ-categorical for all unctbl cardinals κ.

What Morley actually proves (roughly) that for all λ-categorical theories T, their models admit an abstract version of a span operation, which allows for defining independence and basis, and hence extend bijections between bases to isomorphisms. And this is why λ-categoricity implies κ-categoricity for all κ, like in our examples (c)−(e).

In (c), the span of a set U of vertices is the union all the connected components of all $u \in U$. In (d), the span is the linear span, and in (e) the span is the algebraic closure.

What is categoricity useful for?

Łoś–Vaught test (for completeness). Let T be a σ-theory that doesn't have finite models. Then if T is κ-categorical for some $\kappa \geq \max(\aleph_0, |\sigma|)$, then T is complete.

Proof. To show that T is complete, we need to show that for any models $\underline{M}, \underline{N} \models T$ we have $\underline{M} \equiv \underline{N}$. Since both $\underline{M}$ and $\underline{N}$ are infinite, their theories $Th(\underline{M})$ and $Th(\underline{N})$ have infinite models, hence upward Löwenheim–Skolem applies and $\underline{M}' \equiv \underline{M}$ and $\underline{N}' \equiv \underline{N}$ of cardinality κ. But $\underline{M}', \underline{N}' \models T$, so $\underline{M}' \cong \underline{N}'$ by κ-categoricity, in particular $\underline{M}' \equiv \underline{N}'$. Thus $\underline{M} \equiv \underline{M}' \equiv \underline{N}' \equiv \underline{N}$. $\square$

Prop. Algebraically closed fields are infinite, i.e. ACF doesn't have finite models.

Proof. Let $F = \{a_1, \dots, a_n\}$ be a finite field. Then the polynomial $(x-a_1)(x-a_2)\dots(x-a_n)+1$ doesn't have a root, so F is not algebraically closed.

Corollary. The theories

- DLO
- T_k (of k-regular acyclic graphs)
- $VC_{\mathbb{Q}}$
- ACF_p for p prime or 0

are complete (in their respective signatures).